Eigenvalue Analysis of Rectangular Mindlin Plates by Chebyshev Pseudospectral Method

Jinhee Lee*
Department of Mechano-Informatics, Hongik University, Chochiwon, Yeonki-kun, Choongnam 339-701, Korea

A study of free vibration of rectangular Mindlin plates is presented. The analysis is based on the Chebyshev pseudospectral method, which uses test functions that satisfy the boundary conditions as basis functions. The result shows that rapid convergence and accuracy as well as the conceptual simplicity are achieved when the pseudospectral method is applied to the solution of eigenvalue problems. Numerical examples of rectangular Mindlin plates with clamped and simply supported boundary conditions are provided for various aspect ratios and thickness-to-length ratios.

Key Words : Eigenvalue, Mindlin Plate, Pseudospectral Method, Chebyshev Polynomials

Nomenclature

a_{kl}, b_{kl}, c_{kl}	: Expansion coefficients
$A_k, B_k, C_k, F_l, U_l, V_l$	: One-dimensional basis functions
D	: Flexural rigidity
E	: Modulus of elasticity
G	: Shear modulus
h	: Thickness of the plate
$M_x, M_y, M_{xy}, Q_x, Q_y$	: Stress resultants
T_n	: Chebyshev polynomials of the first kind
w, W	: Transverse displacement
X	: Size of the rectangle in x-direction
Y	: Size of the rectangle in y-direction
β	: Shear correction factor
λ_{ij}^2	: Nondimensionalized frequency parameter
ν	: Poisson's ratio
ρ	: Density of the plate
ψ_x, Ψ_x	: Bending rotation normal to the midplane in x-direction
ψ_y, Ψ_y	: Bending rotation normal to the midplane in y-direction
ω	: Natural frequency in [radian/sec]

Subscripts

n	: Normal to the boundary
s	: Tangential to the boundary

E-mail : jinhlee@wow.hongik.ac.kr
TEL : +82-41-860-2589; **FAX :** +82-41-863-0559
Department of Mechano-Informatics, Hongik University, Chochiwon, Yeonki-kun, Choongnam 339-701, Korea. (Manuscript **Received** August 29, 2002; **Revised** November 28, 2002)

1. Introduction

Plate vibration is important in many applications in mechanical, civil and aerospace engineering. Real plates may have appreciable thickness in which case the transverse shear and the rotary inertia are not negligible as assumed in the classical plate theory. As a result the thick plate model based on the Mindlin theory has gained more popularity. In recent years, the eigenvalue analyses of plates based on the Mindlin theory have been extensively investigated and new methods have been proposed.

Research on the Mindlin plate vibration can be divided into three categories. First, there exist exact solutions for a very restricted number of simple cases (Srinivas and Rao, 1970). Second, semi-analytic solutions are available. These cases include the Rayleigh-Ritz method (Dawe and Roufaeil, 1980; Chakraverty et al., 1999) and the differential quadrature method (Bert and Malik, 1996; Liew and Teo, 1999). Finally, there are the most widely used discretization methods such as the finite element method, the finite strip method and the finite difference method as can be found in the following survey articles (Leissa, 1981; Leissa, 1986; Liew et al., 1995).

As it is more useful to have analytical results than to resort to a numerical method, most efforts focus on developing efficient semi-analytic solutions. The pseudospectral method can be considered as a spectral method that performs a collocation process. As the formulation is simple and powerful enough to produce approximate solutions that are close to exact solutions, this method has been used extensively in fluid mechanics research (Pyret and Taylor, 1990). The pseudospectral method can be made as spatially accurate as desired through exponential rate of convergence with mesh refinement. It also permits the choice of a wide variety of functions for the expansion. Since the basis functions can be differentiated analytically and since each spectral coefficient is determined by all the grid point values the pseudospectral rules are N-point formulas, and one would need an N-th order finite difference or finite element method with an error of $O(h^N)$ to equal the accuracy of the pseudospectral procedure with N collocation points (Boyd, 1989).

Even though this method could be used for the solution of structural mechanics problems, it has been largely unnoticed by the structural mechanics community and few articles are available where the pseudospectral method has been applied. For instance spectral element method was applied to the vibration analysis of plates subject to dynamic loads (Lee and Lee, 1998). Chebyshev collocation method was applied to the free vibration analyses of axisymmetric circular plates (Soni and Amba-Rao, 1975) and axisymmetric annular plates (Gupta and Lal, 1985), where fourth order differential equations in terms of ψ were formed by eliminating w. The boundary conditions that does not contain the eigenvalue were combined with the governing equations to form the characteristic equations from which the eigenvalues were calculated. The collocation method along with the power series representation of the dependent variables was also used in the free vibration analysis of the Mindlin plates (Mikami and Yoshimura, 1984). Recently, the pseudospectral method was used in an eigenvalue problem of circular Mindlin plates (Lee, 2002).

In the present work, the pseudospectral method is applied to the free vibration analysis of rectangular plates based on the Mindlin theory.

2. Pseudospectral Formulations

The equations of motion of homogeneous, isotropic plates based on the Mindlin theory are

$$\begin{aligned}
&\frac{\partial M_x}{\partial x}+\frac{\partial M_{xy}}{\partial y}-Q_x=\frac{\rho h^3}{12}\frac{\partial^2 \Psi_x}{\partial t^2}\\
&\frac{\partial M_{xy}}{\partial x}+\frac{\partial M_y}{\partial y}-Q_y=\frac{\rho h^3}{12}\frac{\partial^2 \Psi_y}{\partial t^2}\\
&\frac{\partial Q_x}{\partial x}+\frac{\partial Q_y}{\partial y}=\rho h\frac{\partial^2 W}{\partial t^2}
\end{aligned} \tag{1}$$

M_x, M_y, M_{xy}, Q_x and Q_y are defined by

$$\begin{aligned}
&M_x=D\left(\frac{\partial \Psi_x}{\partial x}+\nu\frac{\partial \Psi_y}{\partial y}\right)\\
&M_y=D\left(\nu\frac{\partial \Psi_x}{\partial x}+\frac{\partial \Psi_y}{\partial y}\right)\\
&M_{xy}=\frac{D(1-\nu)}{2}\left(\frac{\partial \Psi_x}{\partial y}+\frac{\partial \Psi_y}{\partial x}\right)\\
&Q_x=\beta Gh\left(\Psi_x+\frac{\partial W}{\partial x}\right)\\
&Q_y=\beta Gh\left(\Psi_y+\frac{\partial W}{\partial y}\right)
\end{aligned} \tag{2}$$

where $D=Eh^3/12(1-\nu^2)$. The substitution of Eq. (2) into Eq. (1) assuming a harmonic motion in time

$$\begin{aligned}
&\Psi_x(x,\ y,\ t)=\psi_x(x,\ y)\sin\omega t\\
&\Psi_y(x,\ y,\ t)=\psi_y(x,\ y)\sin\omega t\\
&W(x,\ y,\ t)=w(x,\ y)\sin\omega t
\end{aligned} \tag{3}$$

yields

$$D\left(\frac{\partial^2 \psi_x}{\partial x^2}+\frac{1-\nu}{2}\frac{\partial^2 \psi_x}{\partial y^2}+\frac{1+\nu}{2}\frac{\partial^2 \psi_y}{\partial x\partial y}\right)$$
$$-\beta Gh\left(\psi_x+\frac{\partial w}{\partial x}\right)=-\omega^2\frac{\rho h^3}{12}\psi_x$$

$$D\left(\frac{1-\nu}{2}\frac{\partial^2\psi_y}{\partial x^2}+\frac{\partial^2\psi_y}{\partial y^2}+\frac{1+\nu}{2}\frac{\partial^2\psi_x}{\partial x\partial y}\right)-\beta Gh\left(\psi_y+\frac{\partial w}{\partial y}\right)=-\omega^2\frac{\rho h^3}{12}\psi_y \tag{4}$$
$$\beta Gh\left(\frac{\partial\psi_x}{\partial x}+\frac{\partial\psi_y}{\partial y}+\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)=-\omega^2\rho hw$$

When the center of the rectangular plate is placed at the origin and the edges are aligned parallel to the Cartesian coordinate axes, it is convenient to normalize the spatial independent variables as follows

$$\xi=\frac{2x}{X}\in[-1,1] \qquad \eta=\frac{2y}{Y}\in[-1,1] \tag{5}$$

and Eq. (4) is rewritten as

$$2D\left(\frac{2}{X^2}\frac{\partial^2\psi_x}{\partial\xi^2}+\frac{1-\nu}{Y^2}\frac{\partial^2\psi_x}{\partial\eta^2}+\frac{1+\nu}{XY}\frac{\partial^2\psi_y}{\partial\xi\partial\eta}\right)-\beta Gh\left(\psi_x+\frac{2}{X}\frac{\partial w}{\partial\xi}\right)=-\omega^2\frac{\rho h^3}{12}\psi_x$$
$$2D\left(\frac{1-\nu}{X^2}\frac{\partial^2\psi_y}{\partial\xi^2}+\frac{2}{Y^2}\frac{\partial^2\psi_y}{\partial\eta^2}+\frac{1+\nu}{XY}\frac{\partial^2\psi_x}{\partial\xi\partial\eta}\right)-\beta Gh\left(\psi_y+\frac{2}{Y}\frac{\partial w}{\partial\eta}\right)=-\omega^2\frac{\rho h^3}{12}\psi_y \tag{6}$$
$$\beta G\left(\frac{2}{X}\frac{\partial\psi_x}{\partial\xi}+\frac{2}{Y}\frac{\partial\psi_y}{\partial\eta}+\frac{4}{X^2}\frac{\partial^2 w}{\partial\xi^2}+\frac{4}{Y^2}\frac{\partial^2 w}{\partial\eta^2}\right)=-\omega^2\rho w$$

ψ_x, ψ_y and w are represented by the same truncation, and the eigenfunction expansions are given by

$$\psi_x(\xi,\eta)=\sum_{k=1}^{K}\sum_{l=1}^{L}a_{kl}A_k(\xi)U_l(\eta)$$
$$\psi_y(\xi,\eta)=\sum_{k=1}^{K}\sum_{l=1}^{L}b_{kl}B_k(\xi)V_l(\eta) \tag{7}$$
$$w(\xi,\eta)=\sum_{k=1}^{K}\sum_{l=1}^{L}c_{kl}C_k(\xi)F_l(\eta)$$

Clamped boundary conditions (C)

$$\psi_n=0,\ \psi_s=0,\ w=0 \tag{8}$$

and simply supported boundary condition (SS)

$$M_n=0,\ \psi_s=0,\ w=0 \tag{9}$$

are considered in this study.

Complex eigenvalues and spurious roots generally occur when the standard set of Chebyshev polynomials is used as basis functions and the boundary conditions that did not contain eigenvalues are included as side constraints to match the number of unknowns. In order to overcome this difficulty, test functions that satisfy the boundary conditions are used as basis functions and the collocation is performed at the internal points only.

ψ_y and w vanish at $\xi=\pm 1$ for the cases in which the boundary conditions of the two opposing edges which are parallel to the y-axis are given as either clamped-clamped (C-C), or simply supported-simply supported (SS-SS) or clamped at $\xi=-1$, and simply supported at $\xi=1$ (C-SS). The basis functions

$$B_{2p-1}(\xi)=C_{2p-1}(\xi)=T_{2p}(\xi)-T_0(\xi)$$
$$B_{2p}(\xi)=C_{2p}(\xi)=T_{2p+1}(\xi)-T_1(\xi) \tag{10}$$
$$(p=1,2,\cdots)$$

satisfy $\psi_y=0$ and $w=0$ at $\xi=\pm 1$. The basis function $A_k(\xi)$, however, is required to satisfy either $\psi_x=0$ or $M_x=0$ at the ends, and is assumed to be

$$A_{2p-1}(\xi)=T_{2p}(\xi)-T_0(\xi)+d_1\xi^2+d_2\xi$$
$$A_{2p}(\xi)=T_{2p+1}(\xi)-T_1(\xi)+d_3\xi^2+d_4\xi \tag{11}$$
$$(p=1,2,\cdots)$$

The coefficients d_1, d_2, d_3 and d_4 in Eq. (11) that satisfy each of C-C, SS-SS and C-SS boundary conditions are calculated as given in Appendix and are listed in Table 1.

Similar situations occur when the boundary conditions of the two opposing edges parallel to the x-axis are given as one of C-C, SS-SS and C-SS types. The basis functions

$$U_{2q-1}(\eta)=F_{2q-1}(\eta)=T_{2q}(\eta)-T_0(\eta)$$
$$U_{2q}(\eta)=F_{2q}(\eta)=T_{2q+1}(\eta)-T_1(\eta) \tag{12}$$
$$(q=1,2,\cdots)$$

Table 1 Coefficients of the correction term in A_k

	d_1	d_2	d_3	d_4
C-C	0	0	0	0
SS-SS	$-2p^2$	0	0	$-4p(p+1)$
C-SS	$-4p^2/3$	$-4p^2/3$	$-4p(p+1)/3$	$-4p(p+1)/3$

Table 2 Coefficients of the correction term in V_l

	e_1	e_2	e_3	e_4
C-C	0	0	0	0
SS-SS	$-2q^2$	0	0	$-4q(q+1)$
C-SS	$-4q^2/3$	$-4q^2/3$	$-4q(q+1)/3$	$-4q(q+1)/3$

guarantee that ψ_x and w vanish at $\eta=\pm 1$. As in Eq. (11), the basis function $V_l(\eta)$ is assumed to be

$$\begin{aligned} V_{2q-1}(\eta) &= T_{2q}(\eta) - T_0(\eta) + e_1\eta^2 + e_2\eta \\ V_{2q}(\eta) &= T_{2q+1}(\eta) - T_1(\eta) + e_3\eta^2 + e_4\eta \\ &(q=1, 2, \cdots) \end{aligned} \tag{13}$$

and the coefficients e_1, e_2, e_3 and e_4 that satisfy each of C-C, SS-SS and C-SS boundary conditions are listed in Table 2.

Substituting Eq. (7) into Eq. (6) and setting the residuals equal to zero at the Gauss-Chebyshev collocation points (ξ_i, η_j), where ξ_i and η_j are given by

$$\begin{aligned} \xi_i &= -\cos\frac{\pi(2i-1)}{2K} \quad (i=1, 2, \cdots, K) \\ \eta_j &= -\cos\frac{\pi(2j-1)}{2L} \quad (j=1, 2, \cdots, L) \end{aligned} \tag{14}$$

yields

$$\begin{aligned} &\sum_{k=1}^{K}\sum_{l=1}^{L}\Big[a_{kl}\Big\{\frac{2}{X^2}A_k''(\xi_i)U_l(\eta_j)+\frac{1-\nu}{Y^2}A_k(\xi_i)U_l^{**}(\eta_j)-\frac{\beta Gh}{2D}A_k(\xi_i)U_l(\eta_j)\Big\} \\ &\quad +b_{kl}\frac{1+\nu}{XY}B_k'(\xi_i)V_l^{*}(\eta_j)-c_{kl}\frac{\beta Gh}{DX}C_k'(\xi_i)F_l(\eta_j)\Big] \\ &=-\omega^2\frac{\rho h^3}{24D}\sum_{k=1}^{K}\sum_{l=1}^{L}a_{kl}A_k(\xi_i)U_l(\eta_j) \\ &\sum_{k=1}^{K}\sum_{l=1}^{L}\Big[a_{kl}\frac{1+\nu}{XY}A_k'(\xi_i)U_l^{*}(\eta_j)+b_{kl}\Big\{\frac{1-\nu}{X^2}B_k''(\xi_i)V_l(\eta_j) \\ &\quad +\frac{2}{Y^2}B_k(\xi_i)V_l^{**}(\eta_j)-\frac{\beta Gh}{2D}B_k(\xi_i)V_l(\eta_j)\Big\}-c_{kl}\frac{\beta Gh}{DY}C_k(\xi_i)F_l^{*}(\eta_j)\Big] \\ &=-\omega^2\frac{\rho h^3}{24D}\sum_{k=1}^{K}\sum_{l=1}^{L}b_{kl}B_k(\xi_i)V_l(\eta_j) \\ &\sum_{k=1}^{K}\sum_{l=1}^{L}\Big[\frac{a_{kl}}{X}A_k'(\xi_i)U_l(\eta_j)+\frac{b_{kl}}{Y}B_k(\xi_i)V_l^{*}(\eta_j)+c_{kl}\Big\{\frac{2}{X^2}C_k''(\xi_i)F_l(\eta_j) \\ &\quad +\frac{2}{Y^2}C_k(\xi_i)F_l^{**}(\eta_j)\Big\}\Big]=-\omega^2\frac{\rho}{2\beta G}\sum_{k=1}^{K}\sum_{l=1}^{L}c_{kl}C_k(\xi_i)F_l(\eta_j) \end{aligned} \tag{15}$$

where $'$ and * denote the differentiation with respect to ξ and η, respectively.

The pseudospectral algebraic system of the standard matrix form

$$[H]\{f\}=\omega^2[Z]\{f\} \tag{16}$$

is formed from Eq. (15), where the eigenvector $\{f\}$ contains the expansion coefficients

$$\{f\}=\{a_{11}, a_{12}, \cdots, a_{KL}, b_{11}, b_{12}, \cdots, b_{KL}, c_{11}, c_{12}, \cdots, c_{KL}\}^T \tag{17}$$

where T stands for the transpose. The algebraic problem Eq. (15) is solved for the eigenvalues using the Eispack RGG subroutine.

3. Numerical Examples

A preliminary test is run to check the convergence of the pseudospectral method applied to the eigenvalue problem of a Mindlin plate. The eigenvalues of a square plate with thickness-to-length ratio $h/X=0.01$ are computed for different $K\times L$, and the computed results are listed in Table 3 where the eigenvalues based on the classical theory (Blevins, 1979) are also given for comparison. The results show rapid convergence of the pseudospectral method in which the convergence of the lowest 13 eigenvalues to 5 significant digits is achieved with $K\times L=12\times 12$, and the lowest 20 eigenvalues with $K\times L=15\times 15$. Poisson's ratio ν and shear correction factor β are 0.3 and 5/6, respectively, throughout the paper and the numbers given in Tables 3~9 are nondimensionalized frequency parameter λ_{ij}^2 defined by

$$\lambda_{ij}^2=\omega_{ij}\frac{X^2}{\sqrt{D/\rho h}} \tag{18}$$

The eigenvalues are computed with $K\times L=15\times 15$ for various aspect ratios Y/X and thickness-to-length ratios, where the C-C-C-C, SS-SS-SS-SS, SS-C-SS-C (simply supported at $\xi=\pm 1$), C-SS-SS-C (simply supported at $\xi=1$ and $\eta=1$), C-SS-C-C (simply supported at $\eta=1$) and SS-SS-SS-C (clamped at $\eta=-1$) boundary conditions are applied. Nondimensionalized frequency parameters of the 9 lowest eigenvalues for each boundary condition are listed in Tables 4~9, where the numbers in the parentheses represent respective vibration modes.

Tables 4~9 show that the computed eigenvalues are in good agreement with those of the classical theory when h/X is very small, but they deviate considerably as h/X becomes larger. In some cases it is observed that the order of appearance of the vibration modes changes as h/X becomes larger. For example, the vibration modes that correspond to the fifth and sixth eigenvalues with $Y/X=0.4$ and $h/X\leq 0.02$ for the C-C-C-C boundary condition in Table 5 are (51) and (12), which turn out to be (12) and (51) with $h/X=0.05$.

Table 3 Convergence test of the pseudospectral method applied to the free vibration of square plates, nondimensionalized frequency parameter λ_{ij}^2 (SS-SS-SS-SS boundary condition, $\beta=5/6$, $\nu=0.3$, $h/X=0.01$)

mode	$K\times L$									Classical theory
	3×3	4×4	5×5	6×6	8×8	10×10	12×12	15×15	18×18	
1	19.965	20.217	19.729	19.731	19.732	19.732	19.732	19.732	19.732	19.74
2	—	52.020	53.445	49.284	49.304	49.303	49.303	49.303	49.303	49.35
3	—	52.020	53.445	49.284	49.304	49.303	49.303	49.303	49.303	49.35
4	—	81.322	85.029	78.775	78.843	78.842	78.842	78.841	78.842	78.96
5	—	—	111.75	118.08	98.950	98.529	98.517	98.517	98.517	98.70
6	—	—	111.75	118.08	98.950	98.529	98.517	98.517	98.517	98.70
7	—	—	140.14	145.12	128.38	128.01	128.00	128.00	128.00	128.3
8	—	—	140.14	145.12	128.38	128.01	128.00	128.00	128.00	128.3
9	—	—	189.97	205.73	168.54	167.33	167.27	167.27	167.27	167.8
10	—	—	—	208.39	168.54	167.33	167.27	167.27	167.27	167.8
11	—	—	—	208.39	177.70	177.09	177.07	177.07	177.07	177.7
12	—	—	—	232.37	197.53	196.72	196.68	196.68	196.68	197.4
13	—	—	—	232.37	197.53	196.72	196.68	196.68	196.68	197.4
14	—	—	—	286.18	246.24	245.66	245.62	245.63	245.63	246.7
15	—	—	—	286.18	246.24	245.66	245.62	245.63	245.63	246.7
16	—	—	—	358.24	313.76	264.89	255.92	255.41	255.41	256.6
17	—	—	—	—	389.47	264.89	255.92	255.41	255.41	256.6
18	—	—	—	—	389.47	293.77	285.21	284.72	284.72	286.2
19	—	—	—	—	413.72	293.77	285.21	284.72	284.72	286.2
20	—	—	—	—	413.72	314.02	314.00	314.00	314.00	315.8

Table 4 Nondimensionalized frequency parameter λ_{ij}^2 of rectangular plates (SS-SS-SS-SS boundary condi -ion, $\beta=5/6$, $\nu=0.3$, $K\times L=15\times 15$)

Y/X	h/X	1	2	3	4	5	6	7	8	9
		(11)	(21)	(31)	(41)	(12)	(22)	(51)	(32)	(42)
	0.005	71.531	101.12	150.41	219.38	256.31	285.84	307.99	335.05	403.90
2/5	0.01	71.460	100.98	150.10	218.72	255.41	284.72	306.69	333.51	401.67
	0.02	71.180	100.42	148.87	216.14	251.91	280.40	301.69	327.62	393.19
	0.05	69.329	96.811	141.22	200.75	231.49	255.56	273.31	294.69	347.58
		(11)	(21)	(12)	(31)	(22)	(32)	(41)	(13)	(23)
	0.005	32.072	61.668	98.651	110.98	128.23	177.51	179.97	209.53	239.07
2/3	0.01	32.057	61.615	98.517	110.81	128.00	177.07	179.53	208.92	238.29
	0.02	32.001	61.407	97.987	110.14	127.11	175.38	177.78	206.57	235.24
	0.05	31.614	60.017	94.545	105.83	121.44	164.96	167.09	192.42	217.24
		(11)	(21)	(12)	(22)	(31)	(13)	(32)	(23)	(14)
	0.01	19.732	49.303	49.303	78.841	98.517	98.517	128.00	128.00	167.27
1	0.02	19.711	49.170	49.170	78.502	97.987	97.987	127.11	127.11	165.75
	0.05	19.562	48.270	48.270	76.260	94.545	94.545	121.44	121.44	156.38
	0.1	19.065	45.483	45.483	69.794	85.038	85.038	106.68	106.68	133.62

Table 5 Nondimensionalized frequency parameter λ_{ij}^2 of rectangular plates (C-C-C-C boundary condition, $\beta=5/6$, $\nu=0.3$, $K\times L=15\times 15$)

Y/X	h/X	1	2	3	4	5	6	7	8	9
		(11)	(21)	(31)	(41)	(51)	(12)	(22)	(32)	(61)
	0.005	147.59	173.54	220.98	291.11	383.40	393.17	419.94	465.85	497.29
2/5	0.01	147.04	172.81	219.88	289.38	380.65	389.94	416.29	461.44	492.90
	0.02	144.90	169.98	215.72	282.89	370.40	377.85	402.71	445.25	476.86
	0.05	132.39	154.02	193.13	249.02	317.33*	319.05**	336.41	368.83	400.48
						(12)*	(51)**			
		(11)	(21)	(12)	(31)	(22)	(41)	(32)	(13)	(42)
	0.005	60.730	93.766	148.62	149.52	179.33	226.50	231.67	281.41	305.58
2/3	0.01	60.637	93.567	148.15	149.07	178.66	225.56	230.60	279.91	303.83
	0.02	60.274	92.793	146.34	147.34	176.09	221.93	226.55	274.17	297.25
	0.05	57.949	88.019	135.43	137.01	161.24	201.30	204.14	242.65	262.41
		(11)	(21)	(12)	(22)	(31)	(13)	(23)	(32)	(14)
	0.01	35.942	73.239	73.239	107.89	131.13	131.13	164.30	164.30	209.46
1	0.02	35.816	72.783	72.783	106.94	129.81	129.81	162.27	162.27	206.39
	0.05	34.982	69.869	69.869	101.13	121.73	121.73	150.30	150.30	188.52
	0.1	32.524	62.039	62.039	86.949	102.43	102.43	123.89	123.89	150.92

Table 6 Nondimensionalized frequency parameter λ_{ij}^2 of rectangular plates (C-SS-SS-C boundary condition, $\beta=5/6$, $\nu=0.3$, $K\times L=15\times 15$)

Y/X	h/X	1	2	3	4	5	6	7	8	9
		(11)	(21)	(31)	(41)	(12)	(51)	(22)	(32)	(61)
	0.005	105.22	133.36	182.45	252.71	320.95	343.69	349.04	396.64	455.08
2/5	0.01	104.99	133.01	181.83	251.61	319.12	341.76	346.90	393.91	451.82
	0.02	104.07	131.62	179.44	247.39	312.16	334.45	338.77	383.67	439.65
	0.05	98.366	123.22	165.56	223.83	274.51	295.26	296.06	331.30	378.19
		(11)	(21)	(12)	(31)	(22)	(41)	(32)	(13)	(23)
	0.005	44.876	76.507	122.23	129.30	152.39	202.39	203.35	244.08	273.76
2/3	0.01	44.834	76.396	121.96	129.01	151.98	201.72	202.65	243.09	272.51
	0.02	44.667	75.960	120.91	127.88	150.38	199.13	199.93	239.28	267.75
	0.05	43.564	73.168	114.35	120.91	140.70	183.73	184.21	217.36	241.05
		(11)	(21)	(12)	(22)	(13)	(31)	(32)	(23)	(41)
	0.01	27.034	60.448	60.697	92.632	114.26	114.41	145.30	145.60	187.70
1	0.02	26.973	60.180	60.433	92.031	113.39	113.55	143.90	144.21	185.49
	0.05	26.564	58.428	58.705	88.219	107.92	108.10	135.34	135.72	172.22
	0.1	25.283	53.393	53.731	78.176	93.877	94.114	114.98	115.43	142.30

4. Conclusions

The pseudospectral method that employs the modified Chebyshev polynomials as basis functions is applied to the free vibration analysis of rectangular plates based on the Mindlin theory. The formulation as well as coding for computation is fairly straightforward. The results of this study show good agreement with those of the classical plate theory when the thickness-to-length ratio is small but quantitative differences in the natural frequencies exist for thicker plates. The example problem demonstrates a rapid conver-

Table 7 Nondimensionalized frequency parameter λ_{ij}^2 of rectangular plates (SS-C-SS-C boundary condition, $\beta=5/6$, $\nu=0.3$, $K\times L=15\times 15$)

Y/X	h/X	1	2	3	4	5	6	7	8	9
		(11)	(21)	(31)	(41)	(51)	(12)	(22)	(61)	(32)
	0.005	145.30	164.51	201.92	260.65	341.45	391.79	414.49	443.93	453.89
2/5	0.01	144.77	163.84	201.01	259.32	339.42	388.59	410.95	440.77	449.76
	0.02	142.69	161.27	197.53	254.29	331.80	376.59	397.76	429.03	434.49
	0.05	130.42	146.51	178.25	227.33	292.54	316.40	332.84	361.41*	371.02**
									(32)*	(61)**
		(11)	(21)	(31)	(12)	(22)	(41)	(32)	(42)	(51)
	0.005	56.321	78.938	123.08	146.12	169.92	188.93	212.53	275.57	275.62
2/3	0.01	56.241	78.803	122.81	145.67	169.33	188.38	211.69	274.27	274.51
	0.02	55.925	78.275	121.77	143.93	167.09	186.21	208.46	269.32	270.22
	0.05	53.874	74.948	115.38	133.39	153.84	173.41	190.05	241.44*	242.16**
									(13)*	(42)**
		(11)	(21)	(12)	(22)	(31)	(13)	(32)	(23)	(41)
	0.01	28.924	54.672	69.194	94.361	102.00	128.68	139.77	154.20	167.79
1	0.02	28.844	54.462	68.801	93.703	101.38	127.45	138.49	152.51	168.17
	0.05	28.311	53.087	66.254	89.555	97.412	119.84	130.72	142.34	158.24
	0.1	26.668	49.113	59.210	78.813	86.844	101.37	112.06	118.92	134.60
		(11)	(12)	(21)	(13)	(22)	(23)	(31)	(14)	(32)
	0.01	17.365	35.311	45.387	61.959	62.226	88.625	94.045	97.202	109.84
3/2	0.02	17.340	35.212	45.263	61.675	61.970	88.096	93.547	96.542	109.12
	0.05	17.172	34.549	44.430	59.815	60.293	84.716	90.324	92.328	104.59
	0.1	16.623	32.505	41.875	54.468	55.468	75.632	81.170	81.413	92.668
								(14)*	(31)*	
		(11)	(12)	(13)	(14)	(21)	(22)	(23)	(15)	(24)
	0.01	12.132	18.357	27.947	40.712	41.346	46.957	56.114	56.604	68.650
5/2	0.02	12.122	18.333	27.892	40.598	41.250	46.828	55.922	56.390	68.357
	0.05	12.057	18.170	27.519	39.835	40.600	45.960	54.653	54.974	66.447
	0.1	11.835	17.634	26.327	37.482	38.554	43.295	50.780*	50.874**	60.954
								(15)*	(23)*	

Table 8 Nondimensionalized frequency parameter λ_{ij}^2 of rectangular plates (C-SS-C-C boundary condition, $\beta=5/6$, $\nu=0.3$, $K\times L=15\times 15$)

Y/X	h/X	1	2	3	4	5	6	7	8	9
		(11)	(21)	(31)	(41)	(12)	(22)	(51)	(32)	(42)
	0.005	106.96	139.47	194.01	269.88	321.82	352.40	366.38	403.80	476.09
2/5	0.01	106.72	139.07	193.27	268.54	319.98	350.19	364.04	400.91	472.12
	0.02	105.76	137.50	190.44	263.45	312.96	341.82	355.23	390.11	457.52
	0.05	99.844	128.17	174.27	235.61	275.07	297.88	309.45	335.54	386.82
		(11)	(21)	(12)	(31)	(22)	(32)	(41)	(13)	(23)
	0.005	48.141	85.441	123.86	143.82	158.11	214.24	222.28	245.14	277.76
2/3	0.01	48.090	85.288	123.57	143.42	157.64	213.40	221.40	244.14	276.45
	0.02	47.886	84.691	122.48	141.88	155.84	210.19	217.99	240.27	271.44
	0.05	46.558	80.929	115.66	132.57	145.11	191.87	198.36	218.08	243.68

Y/X	h/X	1	2	3	4	5	6	7	8	9
		(11)	(12)	(21)	(22)	(13)	(31)	(23)	(32)	(14)
	0.01	31.794	63.226	70.934	100.53	116.04	129.92	151.34	158.84	188.99
1	0.02	31.698	62.918	70.516	99.747	115.13	128.66	149.74	157.01	186.72
	0.05	31.060	60.923	67.822	94.896	109.39	120.87	140.11	146.05	173.18
	0.1	29.130	55.334	60.457	82.667	94.879	102.09	117.90	121.29	142.88
		(11)	(12)	(13)	(21)	(22)	(14)	(23)	(31)	(24)
	0.01	25.837	38.051	60.212	65.388	77.369	91.922	98.343	124.48	128.67
3/2	0.02	25.770	37.925	59.942	65.030	76.884	91.351	97.609	123.32	127.50
	0.05	25.320	37.091	58.183	62.700	73.787	87.710	93.019	116.11	120.35
	0.1	23.917	34.613	53.198	56.178	65.480	77.996	81.333	98.418	103.14
		(11)	(12)	(13)	(14)	(15)	(21)	(22)	(23)	(16)
	0.01	23.420	26.992	33.751	44.056	57.918	62.852	66.799	73.614	75.248
5/2	0.02	23.364	26.920	33.648	43.900	57.674	62.521	66.428	73.173	74.863
	0.05	22.987	26.437	32.968	42.876	56.083	60.349	64.016	70.337	72.384
	0.1	21.785	24.940	30.915	39.866	51.533	54.198	57.295	62.628	65.531

Table 9 Nondimensionalized frequency parameter λ_{ij}^2 of rectangular plates (SS-SS-SS-C boundary condition, β, 5/6, $\nu=0.3$, $K\times L=15\times 15$)

Y/X	h/X	1	2	3	4	5	6	7	8	9
		(11)	(21)	(31)	(41)	(12)	(51)	(22)	(32)	(61)
	0.005	103.85	128.23	172.20	236.94	320.18	322.43	346.03	390.19	428.37
2/5	0.01	103.62	127.91	171.68	236.03	318.37	320.84	343.94	387.60	425.69
	0.02	102.74	126.65	169.64	232.53	311.46	314.81	336.02	377.83	415.58
	0.05	97.187	118.96	157.59	212.59	274.00	281.91	293.86	327.39	363.14
		(11)	(21)	(31)	(12)	(22)	(41)	(32)	(13)	(42)
	0.005	42.516	68.975	116.20	120.91	147.51	183.94	193.60	243.17	259.85
2/3	0.01	42.479	68.893	115.99	120.65	147.14	183.44	193.00	242.19	258.82
	0.02	42.334	68.568	115.17	119.63	145.69	181.53	190.66	238.43	254.85
	0.05	41.367	66.459	110.04	113.27	136.84	169.95	176.80	216.73	232.24
		(11)	(21)	(12)	(22)	(31)	(13)	(32)	(23)	(41)
	0.01	23.632	51.619	58.566	85.974	100.08	112.95	133.43	140.42	168.42
1	0.02	23.590	51.456	58.328	85.501	99.508	112.12	132.37	139.18	166.86
	0.05	23.306	50.370	56.755	82.451	95.844	106.86	125.78	131.51	157.25
	0.1	22.389	47.104	52.150	74.105	85.876	93.227	109.26	112.74	134.08
		(11)	(12)	(21)	(13)	(22)	(23)	(14)	(31)	(32)
	0.01	15.573	31.051	44.526	55.326	59.391	83.462	88.273	93.512	107.88
3/2	0.02	15.557	30.986	44.413	55.127	59.179	83.038	87.786	93.028	107.22
	0.05	15.445	30.546	43.648	53.809	57.776	80.287	84.622	89.882	103.00
	0.1	15.073	29.145	41.271	49.876	53.620	72.623	75.890	81.129	91.686
		(11)	(12)	(13)	(14)	(21)	(22)	(15)	(23)	(24)
	0.01	11.748	17.181	25.903	37.802	41.175	46.321	52.843	54.814	66.577
5/2	0.02	11.739	17.163	25.861	37.714	41.081	46.199	52.673	54.640	66.318
	0.05	11.683	17.036	25.574	37.116	40.444	45.381	51.536	53.484	64.614
	0.1	11.490	16.612	24.639	35.232	38.430	42.844	48.092	49.989	59.620

gence and accuracy as well as the conceptual simplicity of the pseudospectral method. It is observed that the choice of the basis functions that satisfy the boundary conditions suppress spurious eigenvalues. Numerical examples of thick rectangular plates with clamped and simply supported boundary conditions are provided for various aspect ratios and thickness-to-radius ratios.

References

Bert, C. W. and Malik, M., 1996, "Differential Quadrature Method in Computational Mechanics: A Review," *Applied Mechanics Review*, Vol. 49, No. 1, pp. 1~28.

Blevins, R. D., 1979, *Formulas for Natural Frequency and Mode Shape*, Van Nostrand Reinhold, New York, pp. 258~261.

Boyd, J. P., 1989, *Chebyshev and Fourier Spectral Methods, Lecture notes in engineering 49*, Springer-Verlag, Berlin, pp. 10~11.

Chakraverty, S., Bhat, R. B. and Stiharu, I., 1999, "Recent Research on Vibration of Structures Using Boundary Characteristic Orthogonal Polynomials in Rayleigh-Ritz Method," *The Shock and Vibration Digest*, Vol. 31, No. 3, pp. 187~194.

Dawe, D. J. and Roufaeil, O. L., 1980, "Rayleigh-Ritz Vibration Analysis of Mindlin Plates," *Journal of Sound and Vibration*, Vol. 69, pp. 345~359.

Gupta, U. S. and Lal, R., 1985, "Axisymmetric Vibrations of Polar Orthotropic Mindlin Annular Plates of Variable Thickness," *Journal of Sound and Vibration*, Vol. 98, No. 4, pp. 565~573.

Lee, J., 2002, "Eigenvalue Analysis of Circular Mindlin Plates Using the Pseudospectral Method," *Journal of Korean Society of Mechanical Engineers A* (Korean with English abstract), Vol. 26, No. 6, pp. 1169~1177.

Lee, U. S. and Lee, J. K., 1998, "Vibration Analysis of the Plates Subject to Distributed Dynamic Loads by Using Spectral Element Method," *KSME International Journal*, Vol. 12, No. 4, pp. 565~571.

Leissa, A. W., 1981, "Plate Vibration Research (1976-1980): Complicating Effects, " *Journal of Sound and Vibration*, Vol. 13, No. 10, pp. 19~36.

Leissa, A. W., 1987, "Recent Studies in Plate Vibration (1981-1985): part II, Complicating Effects," *Journal of Sound and Vibration*, Vol. 19, No. 3, pp. 10~24.

Liew, K. M. and Teo, T. M., 1999, "Three-Dimensional Vibration Analysis of Rectangular Plates Based on Differential Quadrature Method," *Journal of Sound and Vibration*, Vol. 220, No. 4, pp. 577~599.

Liew, K. M., Xiang, Y. and Kitipornchai, S., 1995, "Research on Thick Plate Vibration: A Litterature Survey," *Journal of Sound and Vibration*, Vol. 180, No. 1, pp. 163~176.

Mikami, T. and Yoshimura, J., 1984, "Application of the Collocation Method to Vibration Analysis of Rectangular Mindlin Plates," *Computers & Structures*, Vol. 18, No. 3, pp. 425~432.

Pyret, R. and Taylor, T. D., 1990, *Computational Methods for Fluid Flow*, Springer-Verlag, pp. 227~247.

Soni, S. R. and Amba-Rao, C. L., 1975, "On Radially Sysymmetric Vibrations of Orthotropic Non-Uniform Disks Including Shear Deformation," *Journal of Sound and Vibration*, Vol. 42, No. 1, pp. 57~63.

Srinivas, S. and Rao, A. K., 1970, "Vibration and Buckling of Simply Supported Thick Orthotropic Rectangular Plates and Laminates," *International Journal of Solids and Structures*, Vol. 6, pp. 1463~1481.

Appendix

1. The simply supported-simply supported boundary condition (SS-SS) for the two opposing edges that are parallel to the y-axis is

$$\begin{cases} M_x=0,\ \psi_y=0,\ w=0 \text{ at } \xi=-1 \\ M_x=0,\ \psi_y=0,\ w=0 \text{ at } \xi=1 \end{cases} \tag{A1}$$

$\psi_y=0$ and $w=0$ at $\xi=\pm 1$ are satisfied by the condition given in Eq. (10), and the remaining condition is

$$\begin{aligned} M_x\big|_{\xi=\pm1} &= D\left(\frac{2}{X}\frac{\partial\psi_x}{\partial\xi}+\nu\frac{2}{Y}\frac{\partial\psi_y}{\partial\eta}\right)\bigg|_{\xi=\pm1} \\ &= \frac{2D}{X}\frac{\partial\psi_x}{\partial\xi}\bigg|_{\xi=\pm1}=0 \end{aligned} \tag{A2}$$

Using the relationship (7), it is worthwhile to note that

$$\left.\frac{dA_k}{d\xi}\right|_{\xi=\pm1}=0 \quad (k=1, 2, \cdots, K) \tag{A3}$$

is a sufficient condition for the zero-moment condition (A2). Setting the differentiation of the odd numbered terms of $A_k(\xi)$ with respect to ξ equal to zero makes

$$\left.\frac{dA_{2p-1}}{d\xi}\right|_{\xi=\pm1}=\left.\left(\frac{dT_{2p}}{d\xi}+2d_1\xi+d_2\right)\right|_{\xi=\pm1}=0 \quad (p=1, 2, \cdots) \tag{A4}$$

Eq. (A4) is rewritten as

$$\begin{cases} -4p^2-2d_1+d_2=0 \text{ at } \xi=-1 \\ 4p^2+2d_1+d_2=0 \text{ at } \xi=1 \end{cases} \tag{A5}$$

and we have

$$d_1=-2p^2,\ d_2=0 \tag{A6}$$

The differentiation of the even numbered terms with respect to ξ makes

$$\left.\frac{dA_{2p}}{d\xi}\right|_{\xi=\pm1}=\left.\left(\frac{dT_{2p+1}}{d\xi}-1+2d_3\xi+d_4\right)\right|_{\xi=\pm1}=0 \tag{A7}$$

Eq. (A7) is also rewritten as

$$\begin{cases} (2p+1)^2-1-2d_3+d_4=0 \text{ at } \xi=-1 \\ (2p+1)^2-1+2d_3+d_4=0 \text{ at } \xi=1 \end{cases} \tag{A8}$$

from which the constants d_3 and d_4 are found to be

$$d_3=0,\ d_4=-4p(p+1) \tag{A9}$$

2. The clamped-simply supported boundary condition (C-SS) for the two opposing edges that are parallel to the y-axis is

$$\begin{cases} \psi_x=0,\ \psi_y=0,\ w=0 \text{ at } \xi=-1 \\ M_x=0,\ \psi_y=0,\ w=0 \text{ at } \xi=1 \end{cases} \tag{A10}$$

$\psi_y=0$ and $w=0$ at $\xi=\pm1$ are satisfied by the condition given in Eq. (10), and the sufficient condition for the clamped-simply supported boundary condition is

$$\begin{cases} A_k=0 \text{ at } \xi=-1 \\ \dfrac{dA_k}{d\xi}=0 \text{ at } \xi=1 \end{cases} \tag{A11}$$

Using the relationships of Eqs. (11) and (A4), the condition for the odd numbered terms is given by

$$\begin{cases} A_{2p-1}|_{\xi=-1}=(T_{2p}-T_0+d_1\xi^2+d_2\xi)|_{\xi=-1}=d_1-d_2=0 \\ \left.\dfrac{dA_{2p-1}}{d_\xi}\right|_{\xi=1}=\left.\left(\dfrac{dT_{2p}}{d\xi}+2d_1\xi+d_2\right)\right|_{\xi=1}=4p^2+2d_1+d_2=0 \end{cases} \tag{A12}$$

from which we have

$$d_1=d_2=-\frac{4p^2}{3} \tag{A13}$$

For the even numbered terms

$$\begin{cases} A_{2p}|_{\xi=-1}=(T_{2p+1}-T_1+d_3\xi^2+d_4\xi)|_{\xi=-1}=d_3-d_4=0 \\ \left.\dfrac{dA_{2p}}{d\xi}\right|_{\xi=1}=\left.\left(\dfrac{dT_{2p+1}}{d\xi}-1+2d_3\xi=d_4\right)\right|_{\xi=1}=(2p+1)^2-1+2d_3+d_4=0 \end{cases} \tag{A14}$$

from which we have

$$d_3=d_4=-\frac{4p(p+1)}{3} \tag{A15}$$